

CHOW GROUPS OF TENSOR-TRIANGULATED CATEGORIES

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ABSTRACT. We recall P. Balmer's definition of tensor-triangular Chow group for a tensor-triangulated category \mathcal{K} and explore some of its properties. We give a proof that for a suitably nice scheme X it recovers the usual notion of Chow group from algebraic geometry when we put $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$. Furthermore, we identify a class of functors for which tensor-triangular Chow groups behave functorially and show that (for suitably nice schemes) proper push-forward and flat pull-back of algebraic cycles can be interpreted as being induced by their derived functors between the bounded derived categories of the involved schemes. We also compute some examples for stable categories from modular representation theory, where we obtain tensor-triangular cycle groups with torsion coefficients. This illustrates our point of view that tensor-triangular cycles are elements of a certain Grothendieck group, rather than \mathbb{Z} -linear combinations of closed subspaces of some topological space. We finish by extending Balmer's definition to the relative setting, where a tensor-triangulated category acts on a triangulated one, which leads us to the notion of relative tensor-triangular Chow groups.

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1. INTRODUCTION

A basic topic in algebraic geometry is the study of algebraic cycles on a variety X under the equivalence relation of rational equivalence. This is usually formalised by the Chow group

$$\mathrm{CH}(X) = \bigoplus_p \mathrm{CH}^p(X)$$

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where $\mathrm{CH}^p(X)$ is the free abelian group on subvarieties $Y \subset X$ of codimension p , modulo the subgroup of cycles rationally equivalent to zero (i.e. those that appear as the divisor of a rational function on a subvariety of codimension $p - 1$).

A question that arises is how to approach the subject from the point of view of the derived category of X . In [5], it is shown that we can reconstruct X from its derived category of perfect complexes $\mathrm{D}^{\mathrm{perf}}(X)$ considered as a *tensor-triangulated category*. Thus, it should also be possible to reconstruct $\mathrm{CH}(X)$ from $\mathrm{D}^{\mathrm{perf}}(X)$ “in purely categorical terms”. More precisely one would like to construct a functor $\mathrm{CH}_p^\Delta(-)$, that takes a tensor-triangulated category \mathcal{K} and produces a group $\mathrm{CH}_p^\Delta(\mathcal{K})$ such that $\mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(X)) \cong \mathrm{CH}^p(X)$.

In this paper we show that such a construction is possible by giving a definition of $\mathrm{CH}_p^\Delta(-)$, suggested to the author by P. Balmer in 2011 and available in [1] (in slightly different form). The essential point is that when one filters the category $\mathrm{D}^{\mathrm{perf}}(X)$ by codimension of support, the successive subquotients split as a coproduct of “local categories” (cf. [6]), analogously to what happens when one performs the same procedure for the abelian category $\mathrm{Coh}(X)$. One then continues to define the codimension- p cycle group of $\mathrm{D}^{\mathrm{perf}}(X)$ as the Grothendieck group of the p -th subquotient of the filtration, keeping in mind Quillen’s work [20, §7]. Finally, one obtains a definition of the codimension- p Chow group of $\mathrm{D}^{\mathrm{perf}}(X)$ by analogy from Quillen’s coniveau spectral sequence. We prove:

Theorem (3.4). *Let X be a non-singular scheme of finite type over a field. Endow $\mathrm{D}^{\mathrm{perf}}(X)$ with the opposite of the Krull codimension as a dimension function (cf. Definition 2.3). Then for all $p \in \mathbb{Z}$,*

$$\mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(X)) \cong \mathrm{CH}^{-p}(X)$$

We also show that $\mathrm{CH}_p^\Delta(-)$ is functorial for the class of exact functors with a relative dimension (cf. Definition 4.1.1). We have

Theorem (4.1.3). *Let $F : \mathcal{K} \rightarrow \mathcal{L}$ be a functor of relative dimension n . Then for all $p \in \mathbb{Z}$, F induces a group homomorphism*

$$\mathrm{CH}_p^\Delta(F) : \mathrm{CH}_p^\Delta(\mathcal{K}) \rightarrow \mathrm{CH}_{p+n}^\Delta(\mathcal{L})$$

and we prove that the proper push-forward and flat pull-back morphisms on the classical Chow groups can be interpreted as special cases of the above theorem.

We then proceed to compute some examples from modular representation theory: the stable module category $kG\text{-stab}$ for a finite group G and a field k whose characteristic divides $|G|$ is a tensor triangulated category and we compute the associated tensor-triangular Chow groups for $G = \mathbb{Z}/p^n\mathbb{Z}$ and $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

Theorem (5.2.1, 5.3.6 and 5.3.8). *Let k be a field with $\mathrm{char}(k) = p$. For $G = \mathbb{Z}/p^n\mathbb{Z}$, we have*

$$\mathrm{CH}_i^\Delta(kG\text{-stab}) = 0 \quad \forall i \neq 0 \quad \mathrm{CH}_0^\Delta(kG\text{-stab}) \cong \mathbb{Z}/p^n\mathbb{Z}$$

and if $p = 2$ and $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ then

$$\mathrm{CH}_i^\Delta(kH\text{-stab}) = 0 \quad \forall i \neq 0, 1 \quad \mathrm{CH}_0^\Delta(kH\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z} \quad \mathrm{CH}_1^\Delta(kH\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z}$$

when we endow $kG\text{-stab}, kH\text{-stab}$ with the Krull dimension as a dimension function and assume that k is algebraically closed for the last isomorphism.

In particular, we see that we get cycle groups with torsion coefficients, which contrasts with the situation in the algebro-geometric case. This illustrates that we view a general cycle, rather than as a \mathbb{Z} -linear combination of irreducible subspaces of codimension p of the spectrum $\mathrm{Spc}(\mathcal{K})$, as an element of a Grothendieck group $K_0(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})$. Only in the non-singular algebro-geometric

examples does this produce coefficients in \mathbb{Z} , due to the “coincidence” that the Grothendieck group of the derived category of finite-length modules over a local ring is isomorphic to \mathbb{Z} .

Finally, we extend our definition to the relative case, i.e. we define the Chow group of a (non-essentially small) compactly generated triangulated category \mathcal{K} relative to the action of a (non-essentially small) tensor-triangulated category \mathcal{T} , as developed by Stevenson [23]. This is motivated by a generalisation of the main result of [6] to the relative case, also due to Stevenson [24]. We show that when one considers the full derived category of complexes of \mathcal{O}_X -modules with quasi-coherent cohomology $D_{\text{Qcoh}}(X)$ on a noetherian scheme X , acting on itself via the left-derived tensor product, we recover the tensor-triangular Chow groups of $D^{\text{perf}}(X)$. This is obtained as an immediate consequence of the following more abstract result:

Theorem (6.2.3). *Let \mathcal{T} be a compactly-rigidly generated tensor-triangulated category with arbitrary set-indexed coproducts. Consider the action of \mathcal{T} on itself via its tensor product, and assume that the local-to-global principle (cf. Definition 6.1.1) holds for this action. Then we have isomorphisms*

$$Z_p^\Delta(\mathcal{T}, \mathcal{T}) \cong Z_p^\Delta(\mathcal{T}^c) \quad \text{and} \quad \text{CH}_p^\Delta(\mathcal{T}, \mathcal{T}) \cong \text{CH}_p^\Delta(\mathcal{T}^c).$$

where \mathcal{T}^c denotes the subcategory of compact objects of \mathcal{T} .

2. DEFINITIONS AND CONVENTIONS

We now recall a couple of facts and definitions needed to define tensor-triangular Chow groups.

Convention 2.1. In the following (until section 6), \mathcal{K} will always denote a *tensor-triangulated category*. By this, we mean an essentially small triangulated category with a compatible symmetric monoidal structure, i.e. the functor $\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ given by the monoidal product should be exact in both variables. We also demand that \mathcal{K} is *rigid*, meaning that there exists an exact functor $D : \mathcal{K}^{\text{op}} \rightarrow \mathcal{K}$ and a natural isomorphism $\text{Hom}_{\mathcal{K}}(a \otimes b, c) \cong \text{Hom}_{\mathcal{K}}(b, D(a) \otimes c)$ for all objects a, b, c in \mathcal{K} . We will not use this property of \mathcal{K} explicitly, but it is needed for some results of [6] that we do use. By a *thick \otimes -ideal* in \mathcal{K} we mean a full triangulated subcategory $\mathcal{L} \subset \mathcal{K}$ such that \mathcal{L} is thick and such that for all $a, b \in \mathcal{K}$ we have $a \in \mathcal{K}, b \in \mathcal{L} \Rightarrow a \otimes b \in \mathcal{L}$. A *prime ideal* $\mathcal{P} \subset \mathcal{K}$ is a thick \otimes -ideal that is a proper subcategory, such that for all $a, b \in \mathcal{K}$ we have $a \otimes b \in \mathcal{P} \Rightarrow a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Following [5], the *spectrum* of \mathcal{K} , denoted by $\text{Spc}(\mathcal{K})$, is the topological space whose underlying set is given by all prime ideals of \mathcal{K} , topologised in the following way: the *support* of an object $a \in \mathcal{K}$ is defined as $\text{supp}(a) := \{\mathcal{P} \in \text{Spc}(\mathcal{K}) \mid a \notin \mathcal{P}\}$. We take all subsets of the form $\text{supp}(a)$ for $a \in \mathcal{K}$ as a basis of closed sets for $\text{Spc}(\mathcal{K})$.

Typical examples of tensor-triangulated categories are $\mathcal{K} = K^b(R - \text{proj})$, the homotopy category of finitely generated, projective modules over a commutative ring R , or more generally $D^{\text{perf}}(X)$, the derived category of perfect complexes on a noetherian scheme X . Later on, we will also look at the case where $\mathcal{K} = kG\text{-stab}$, the stable module category of a finite group G over a field k such that $\text{char}(k)$ divides $|G|$. In these examples, the spectrum of the categories is known: $\text{Spc}(D^{\text{perf}}(X)) \cong X$ and $\text{Spc}(kG\text{-stab}) \cong \mathcal{V}_G(k)$, the projective support variety of kG (cf. [5, 6]).

Convention 2.2. The idempotent completion (a.k.a. idempotent splitting, Cauchy completion, Karoubi envelope) of a triangulated category \mathcal{L} will be denoted by \mathcal{L}^{\natural} . The category \mathcal{L}^{\natural} can be endowed with an essentially unique triangulated structure such that the embedding $\iota : \mathcal{L} \hookrightarrow \mathcal{L}^{\natural}$ is exact, and that every exact functor $\mathcal{L} \rightarrow \mathcal{K}$ with \mathcal{K} idempotent complete factors uniquely up to natural equivalence through \mathcal{L}^{\natural} over ι . In the sequel \mathcal{L}^{\natural} will always be implicitly assumed to have this triangulation (for more information on this, see [4]).

The Chow groups to be defined will depend on a notion of dimension. Recall the following definition from [6]:

Definition 2.3. A *dimension function* on \mathcal{K} is a map $\dim : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathbb{Z} \cup \{\pm\infty\}$ such that the following two conditions hold:

- (1) If $\mathcal{Q} \subset \mathcal{P}$ are thick tensor-ideals of \mathcal{K} , then $\dim(\mathcal{Q}) \leq \dim(\mathcal{P})$.
- (2) If $\mathcal{Q} \subset \mathcal{P}$ and $\dim(\mathcal{Q}) = \dim(\mathcal{P}) < \infty$, then $\mathcal{Q} = \mathcal{P}$.

For a subset $V \subset \mathrm{Spc}(\mathcal{K})$, we define $\dim(V) := \sup\{\dim(\mathcal{P}) \mid \mathcal{P} \in V\}$. For every $p \in \mathbb{Z} \cup \{\pm\infty\}$, we define the full subcategory $\mathcal{K}_{(p)} := \{a \in \mathcal{K} : \dim(\mathrm{supp}(a)) \leq p\}$.

Remark 2.4. From the properties of $\mathrm{supp}(-)$, it follows that $\mathcal{K}_{(p)}$ is a thick tensor ideal in \mathcal{K} .

Example 2.5. The main examples of dimension functions considered here are the Krull dimension and the opposite of the Krull codimension. For $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, its *Krull dimension* $\dim_{\mathrm{Krull}}(\mathcal{P})$ is the maximal length n of a chain of irreducible closed subsets

$$\emptyset \subsetneq C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_n = \overline{\{\mathcal{P}\}}.$$

Dually, we define the *opposite of the Krull codimension* $-\mathrm{codim}_{\mathrm{Krull}}(\mathcal{P})$ as follows: if we have a chain of irreducible closed subsets of maximal length

$$\overline{\{\mathcal{P}\}} = C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_n = \text{maximal irreducible component of } \mathrm{Spc}(\mathcal{K}) \text{ containing } \mathcal{P}$$

we set $-\mathrm{codim}_{\mathrm{Krull}}(\mathcal{P}) = -n$.

Remark 2.6. One can show that $\mathrm{Spc}(\mathcal{K})$ is always a *spectral* topological space, i.e. it is homeomorphic to the spectrum of a commutative ring (cf. [7, Proposition 11]). This might give an indication of why these examples are useful.

We are now ready to give a definition of tensor-triangular cycle groups and Chow groups, following the ideas from [1].

Definition 2.7. Let \mathcal{K} be a tensor-triangulated category equipped with a dimension function \dim . For $p \in \mathbb{Z}$ we define the *p-dimensional cycle group* of \mathcal{K} as

$$Z_p^\Delta(\mathcal{K}) := K_0((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural).$$

We also need a notion of generalised rational equivalence, which we describe next. Look at the following diagram of subcategories and sub-quotients of \mathcal{K}

$$\begin{array}{ccc} \mathcal{K}_{(p)} & \xrightarrow{I} & \mathcal{K}_{(p+1)} \\ \downarrow Q & & \\ \mathcal{K}_{(p)}/\mathcal{K}_{(p-1)} & \xrightarrow{J} & (\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural \end{array}$$

where I, J denote the obvious embeddings and Q is the Verdier quotient functor. After applying K_0 we get a diagram

$$\begin{array}{ccc} K_0(\mathcal{K}_{(p)}) & \xrightarrow{i} & K_0(\mathcal{K}_{(p+1)}) \\ \downarrow q & & \\ K_0(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}) & \xrightarrow{j} & K_0((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural) = Z_p^\Delta(\mathcal{K}) \end{array}$$

where the lowercase maps are induced by the uppercase functors.

Definition 2.8. Let \mathcal{K} be a tensor-triangulated category equipped with a dimension function \dim . For $p \in \mathbb{Z}$ we define the p -dimensional Chow group of \mathcal{K} as

$$\mathrm{CH}_p^\Delta(\mathcal{K}) := \mathrm{Z}_p^\Delta(\mathcal{K})/j \circ q(\ker(i)).$$

Remark 2.9. It may not be immediately obvious to the reader how the above definitions are motivated. The following account might remedy the situation: assume that $\mathrm{Spc}(\mathcal{K})$ is a noetherian topological space. Following [6], the quotient functors $Q_P : \mathcal{K} \rightarrow \mathcal{K}/P$ for $P \in \mathrm{Spc}(\mathcal{K})$ induce an exact equivalence

$$(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural \xrightarrow{\sim} \coprod_{P \in \mathrm{Spc}(\mathcal{K})_p} \mathrm{Min}(\mathcal{K}_P)$$

where $\mathrm{Spc}(\mathcal{K})_p$ denotes the set of points P in $\mathrm{Spc}(\mathcal{K})_p$ that have dimension p , where \mathcal{K}_P is the local category $(\mathcal{K}/P)^\natural$ and where $\mathrm{Min}(\mathcal{K}_P)$ is the full subcategory of objects with minimal support. (In [6] this subcategory is denoted by $\mathrm{FL}(\mathcal{K}_P)$; this decomposition is the reason why we idempotent-complete the quotient $\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$.) In analogy with the theory of algebraic cycles, an element of the p -dimensional tensor-triangular cycle group of \mathcal{K}

$$\mathrm{Z}_p^\Delta(\mathcal{K}) = \mathrm{K}_0((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural) \cong \coprod_{P \in \mathrm{Spc}(\mathcal{K})_p} \mathrm{K}_0(\mathrm{Min}(\mathcal{K}_P))$$

can thus be regarded as a sum of p -dimensional (relative to the dimension function!) irreducible closed subsets of $\mathrm{Spc}(\mathcal{K})$ with coefficients in $\mathrm{K}_0(\mathrm{Min}(\mathcal{K}_P))$. In the case that $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ for X a non-singular noetherian scheme, one can show that this Grothendieck group is isomorphic to \mathbb{Z} as follows: the category \mathcal{K}_P is equivalent to $\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,\rho(P)}\text{-mod})$, the derived category of perfect complexes of $\mathcal{O}_{X,\rho(P)}$ -modules where ρ denotes the isomorphism $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \rightarrow X$. As X is non-singular there is another equivalence $\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,\rho(P)}\text{-mod}) \cong \mathrm{D}^b(\mathcal{O}_{X,\rho(P)}\text{-mod})$. Next we observe that $\mathrm{Min}(\mathrm{D}^b(\mathcal{O}_{X,\rho(P)}\text{-mod}))$ is the subcategory of complexes with finite length homology, which in turn is equivalent to $\mathrm{D}^b(\mathcal{O}_{X,\rho(P)}\text{-mod-fl})$, the bounded derived category of finite length modules over $\mathcal{O}_{X,\rho(P)}$. Summarizing, we have

$$\mathrm{K}_0(\mathrm{Min}(\mathcal{K}_P)) \cong \mathrm{K}_0(\mathrm{D}^b(\mathcal{O}_{X,\rho(P)}\text{-mod-fl})) \cong \mathrm{K}_0(\mathcal{O}_{X,\rho(P)}\text{-mod-fl}) \cong \mathbb{Z}$$

where the last isomorphism is induced by the length function. Thus we recover our usual notion of cycle group in this case.

Remark 2.10. Following [1], we can push the analogy of the previous remark even further and define “divisors of functions” in order to obtain tensor-triangular Chow groups: for an object $a \in \mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$ and an automorphism $f : a \rightarrow a$ in $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$, choose a fraction $a \xrightarrow{\alpha} b \xleftarrow{\beta} a$ in $\mathcal{K}_{(p+1)}$ representing f . Set

$$\mathrm{div}^\Delta(f) := \left([Q_P^\natural(\mathrm{cone}(\alpha))] - [Q_P^\natural(\mathrm{cone}(\beta))] \right)_{P \in \mathrm{Spc}(\mathcal{K})_p} \in \coprod_{P \in \mathrm{Spc}(\mathcal{K})_p} \mathrm{K}_0(\mathrm{Min}(\mathcal{K}_P))$$

where $Q_P^\natural = I_P \circ Q_P$ and $I_P : \mathcal{K}/P \rightarrow \mathcal{K}_P$ is the embedding into the idempotent completion. An application of the octahedral axiom shows that $\mathrm{div}^\Delta(f)$ does not depend on the choice of α and β : indeed, if we have an equivalent fraction $a \xrightarrow{\alpha'} c \xleftarrow{\beta'} a$, there is by definition a commutative

diagram in $\mathcal{K}_{(p+1)}$

$$\begin{array}{ccccc}
 & & b & & \\
 & \nearrow \alpha & \downarrow f & \nwarrow \beta & \\
 a & \longrightarrow & d & \longleftarrow & a \\
 & \searrow \alpha' & \uparrow g & \swarrow \beta' & \\
 & & c & &
 \end{array}$$

Using the octahedral axiom, this shows that both $[\text{cone}(\alpha)] - [\text{cone}(\alpha')]$ and $[\text{cone}(\beta)] - [\text{cone}(\beta')]$ are equal to $[\text{cone}(f)] - [\text{cone}(g)]$ in $K_0(\mathcal{K}_{(p+1)})$ and so we certainly have

$$[\text{cone}(\alpha)] - [\text{cone}(\beta)] = [\text{cone}(\alpha')] - [\text{cone}(\beta')]$$

in $K_0(\mathcal{K})$. This also implies that

$$[Q_P^\natural(\text{cone}(\alpha))] - [Q_P^\natural(\text{cone}(\beta))] = [Q_P^\natural(\text{cone}(\alpha'))] - [Q_P^\natural(\text{cone}(\beta'))]$$

for all $P \in \text{Spc}(\mathcal{K})_p$ and thus shows the desired independence.

We now define our new Chow group as “cycles modulo divisors of functions”,

$$\text{ch}_p^\Delta(\mathcal{K}) := \left(\coprod_{P \in \text{Spc}(\mathcal{K})_p} K_0(\text{Min}(\mathcal{K}_P)) \right) / \mathfrak{I}$$

where \mathfrak{I} is the subgroup generated by all the $\text{div}^\Delta(f)$. It is not difficult to see that $\mathfrak{I} \subset j_* q(\ker(i))$ under the identification $K_0((\mathcal{K}_p)/\mathcal{K}_{(p-1)})^\natural \cong \coprod_{P \in \text{Spc}(\mathcal{K})_p} K_0(\text{Min}(\mathcal{K}_P))$, but it is not clear to the author if the other inclusion holds in general. Thus, this procedure gives us a potentially *inequivalent* definition of tensor-triangular Chow groups. However, there will always exist an epimorphism

$$\text{ch}_p^\Delta(\mathcal{K}) \rightarrow \text{CH}_p^\Delta(\mathcal{K})$$

for all p . We will see later on (cf. Remark 3.8) that the two notions coincide when we are dealing with separated, non-singular schemes of finite type over a field that have an ample line bundle.

3. AGREEMENT WITH ALGEBRAIC GEOMETRY

We want to show now that the tensor-triangular Chow groups carry their name for a reason. As we will see, they are — at least in the non-singular case — an honest generalisation of the classical Chow groups from algebraic geometry.

Convention 3.1. We now fix some notation for the rest of the section: if not explicitly stated otherwise, let X denote a separated, non-singular scheme of finite type over a field, and $\text{D}^{\text{perf}}(X)$ be the derived category of perfect complexes of \mathcal{O}_X -modules, which is equivalent to $\text{D}^b(X)$, the bounded derived category of coherent sheaves on X . We will also assume that $\text{D}^{\text{perf}}(X)$ is equipped with $-\text{codim}_{\text{Knull}}$ as a dimension function.

In order to proceed, it is necessary to use some higher algebraic K-theory as developed by Quillen. We recall the following material from [20, §7]: Consider the abelian category $\text{Coh}(X)$ of coherent sheaves on X . There is a filtration of this category by codimension of support:

$$\dots \subset M^i \subset M^{i-1} \subset \dots \subset M^0 = \text{Coh}(X)$$

where M^p denotes the Serre subcategory of coherent sheaves whose codimension of support is $\geq p$. Thus, for every p , there is an exact sequence of abelian categories

$$M^{p+1} \hookrightarrow M^p \twoheadrightarrow M^p/M^{p+1}$$

which induces a long exact localisation sequence of K-groups

$$(1) \quad \cdots \longrightarrow K_j(M^{p+1}) \xrightarrow{i_j^p} K_j(M^p) \xrightarrow{q_j^p} K_j(M^p/M^{p+1}) \xrightarrow{b_j^p} K_{j-1}(M^{p+1}) \xrightarrow{i_{j-1}^p} K_{j-1}(M^p) \rightarrow \cdots$$

Combining these long exact sequences for all p , we can form the associated exact couple and obtain the Quillen coniveau spectral sequence as in [20, §7, Theorem 5.4] with E_1 -page

$$E_1^{p,q} = K_{-p-q}(M^p/M^{p+1}).$$

We are especially interested in the boundary map

$$(2) \quad d_1 : K_1(M^{s-1}/M^s) \xrightarrow{b_1^{s-1}} K_0(M^s) \xrightarrow{q_0^s} K_0(M^s/M^{s+1})$$

of this spectral sequence. Using that $K_i(M^s/M^{s+1}) \cong \coprod_{x \in X^{(s)}} K_i(k(x))$, where $X^{(s)}$ denotes the set of points of X whose closure has codimension s in X , Quillen proves the following:

Theorem 3.2 (cf. [20, §7, Proposition 5.14]). *The image of*

$$d_1 : K_1(M^{s-1}/M^s) \longrightarrow K_0(M^s/M^{s+1}) \cong \coprod_{x \in X^{(s)}} K_0(k(x)) \cong \coprod_{x \in X^{(s)}} \mathbb{Z} = Z^s(X)$$

is the subgroup of codimension- p cycles rationally equivalent to zero. In other words, $\text{coker}(d_1) \cong \text{CH}^s(X)$.

Now we pass to our setting, where we work with the triangulated category $\text{D}^{\text{perf}}(X) \cong \text{D}^b(X)$ instead of the abelian category $\text{Coh}(X)$. Recall that the defining diagram for the tensor-triangular Chow groups in this case is given as follows:

$$\begin{array}{ccc} K_0(\text{D}^b(X)_{(p)}) & \xrightarrow{i} & K_0(\text{D}^b(X)_{(p+1)}) \\ \downarrow q & & \\ K_0(\text{D}^b(X)_{(p)}/\text{D}^b(X)_{(p-1)}) & & \\ \downarrow j & & \\ K_0((\text{D}^b(X)_{(p)}/\text{D}^b(X)_{(p-1)})^\sharp) = Z_p^\Delta(\mathcal{K}) & & \end{array}$$

This diagram maps to a similar one involving the related abelian categories:

$$(3) \quad \begin{array}{ccccc} K_0(\text{D}^b(X)_{(p)}) & \xrightarrow{i} & K_0(\text{D}^b(X)_{(p+1)}) & & \\ \downarrow q & \searrow & \downarrow & \searrow & \\ K_0(\text{D}^b(X)_{(p)}/\text{D}^b(X)_{(p-1)}) & & K_0(M^{-p}) & \xrightarrow{i_0} & K_0(M^{-p-1}) \\ \downarrow j & \searrow & \downarrow q_0 & & \\ K_0((\text{D}^b(X)_{(p)}/\text{D}^b(X)_{(p-1)})^\sharp) = Z_p^\Delta(\mathcal{K}) & & K_0(M^{-p}/M^{-p+1}) & & \end{array}$$

The diagonal homomorphisms are all given by the formula $[C^\bullet] \mapsto \sum_i (-1)^i [H^i(C^\bullet)]$. We proceed to show that these are actually all isomorphisms, which follows from the standard fact that there are exact equivalences

$$(4) \quad \text{D}^b(X)_{(q)} \cong \text{D}^b(M^{-q})$$

and

$$(5) \quad \text{D}^b(X)_{(p)}/\text{D}^b(X)_{(p-1)} \cong \text{D}^b(M^{-p}/M^{-p+1})$$

for all $q \in \mathbb{Z}$. Indeed, the diagonal maps are then just the usual isomorphisms between $K_0(D^b(\mathcal{A}))$ and $K_0(\mathcal{A})$ for some abelian category \mathcal{A} . This also proves that j is the identity morphism, as the derived category of an abelian category is idempotent complete [4, Corollary 2.10].

The proof of the equivalences (4) and (5) is a direct consequence of [17, Section 1.15] and the following lemma.

Lemma 3.3. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence in $\text{Coh}(X)$. Then there exist coherent sheaves A', A'' on X with $\text{supp}(A'), \text{supp}(A'') \subset \text{supp}(A)$ that fit into a commutative diagram with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow f & & \downarrow g \\ 0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' \longrightarrow 0 \end{array}$$

Proof. Suppose that A is supported on a closed subscheme with associated ideal sheaf I . As X is noetherian, we can use the sheaf-theoretic version of the Artin-Rees lemma (cf. [22, Lemma 25.12.3]) which says that there exists a $c > 0$ such that for all $n > c$ we have $I^{n-c}(I^c A \cap B) = I^n A$. Now take some n_0 such that $I^{n_0-c} A = 0$, then we get the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & B/I^{n_0} B & \longrightarrow & C/(I^{n_0} C) \longrightarrow 0 \end{array}$$

where the vertical arrows are given by the canonical projections. It is easy to see that the diagram commutes and that all sheaves in the lower row have their support contained in $\text{supp}(A)$. \square

As we know now that the diagonal maps in diagram (3) are isomorphisms and that j is the identity morphism we see that $j \circ q(\ker(i)) \cong q_0(\ker(i_0)) = q_0(\text{im}(b_0)) = \text{im}(d_1)$ (see Theorem 3.2). We have now proved the following:

Theorem 3.4. *Let X be a separated, non-singular scheme of finite type over a field and assume that the tensor-triangulated category $D^{\text{perf}}(X)$ is equipped with the dimension function $-\text{codim}_{\text{Kfull}}$. Then there are isomorphisms*

$$Z_p^\Delta(D^{\text{perf}}(X)) \cong Z^{-p}(X) \quad \text{and} \quad \text{CH}_p^\Delta(D^{\text{perf}}(X)) \cong \text{CH}^{-p}(X)$$

\square

A couple of remarks are in order:

Remark 3.5. One can also produce a more “high-level” proof of the above theorem using Waldhausen models for the categories $D^{\text{perf}}(X)_{(p)}$. This makes it possible to define $K_i(D^{\text{perf}}(X)_{(p)})$ for $i > 0$ as in [26]. Then one obtains long exact localisation sequences which are isomorphic to the sequences (1) and we get a new spectral sequence which is isomorphic to Quillen’s coniveau spectral sequence. In particular, we can talk about the cokernel of the map d_1 (as in (2)) in this new spectral sequence which is then isomorphic to the cokernel of d_1 in Quillen’s coniveau spectral sequence.

Remark 3.6. As we have already seen in Remark 2.9, we don’t need the isomorphism

$$K_0(D^{\text{perf}}(X)_{(p)}/D^{\text{perf}}(X)_{(p-1)}) \cong K_0(M^{-p}/M^{-p+1})$$

to show $Z_p^\Delta(D^{\text{perf}}(X)) \cong Z^{-p}(X)$.

Remark 3.7. The isomorphism $K_0(M^p/M^{p+1}) \xrightarrow{\sim} Z^p(X)$ is explicitly given as follows: for an object $a \in K_0(M^p/M^{p+1})$, set

$$\sigma([a]) = \sum_{Q \in X^{(p)}} \text{length}_{\mathcal{O}_{X,Q}}(a_Q) \cdot \overline{\{Q\}}.$$

(Recall that the objects of M^p/M^{p+1} are the same as the objects of M^p , so it makes sense to localise a at Q). From this we deduce the following explicit formula for the isomorphism

$$\rho_X : Z_p^\Delta(D^{\text{perf}}(X)) \xrightarrow{\sim} Z^{-p}(X).$$

For

$$[C^\bullet] \in Z_p^\Delta(D^{\text{perf}}(X)) = K_0(D^{\text{perf}}(X)_{(p)} / D^{\text{perf}}(X)_{(p-1)})$$

we set

$$\rho_X([C^\bullet]) = \sum_i \sum_{Q \in X^{(-p)}} (-1)^i \text{length}_{\mathcal{O}_{X,Q}} \left(H^i(C^\bullet)_Q \right) \cdot \overline{\{Q\}}.$$

The proof of Theorem 3.4 shows that ρ_X factors through $\text{CH}_p^\Delta(D^{\text{perf}}(X))$ and by abuse of notation, we shall denote the induced isomorphism $\text{CH}_p^\Delta(D^{\text{perf}}(X)) \rightarrow \text{CH}^{-p}(X)$ by ρ_X as well.

Remark 3.8. We can also show that the alternative Chow groups $\text{ch}_p^\Delta(D^{\text{perf}}(X))$ defined in Remark 2.10 coincide with our usual definition in this example, when we assume that X has an ample line bundle \mathcal{L} . Using Theorem 3.4 and the discussion from Remark 2.10, we already know that the subgroup \mathfrak{I} is contained in the subgroup of cycles rationally equivalent to zero. Thus, it suffices to show that any cycle rationally equivalent to zero can be obtained as $\text{div}^\Delta(f)$ for some $f \in \text{Aut}(a), a \in D^b(X)_{(p+1)} / D^b(X)_{(p)}$. The essential point is that for a subvariety $V \subset X$ of codimension $-(p+1)$ we can write the function field of V as $k(V) = \left(\bigoplus_{i \geq 0} \Gamma(X, \mathcal{O}_V \otimes \mathcal{L}^{\otimes i}) \right)_{((0))}$, where $\mathcal{O}_V := \mathcal{O}_X / \mathcal{I}_V$ and \mathcal{I}_V is the ideal sheaf associated to V . Indeed, this is a consequence of [15, Théorème 4.5.2] and the fact that the restriction of an ample line bundle to a closed subscheme is ample.

Thus, for $h \in k(V)$, we can write $h = f/g$ with $f, g \in \Gamma(X, \mathcal{O}_V \otimes \mathcal{L}^{\otimes n})$ for some $n \in \mathbb{N}$. From this, we obtain exact sequences

$$0 \rightarrow \mathcal{O}_V \xrightarrow{m_f} \mathcal{O}_V \otimes \mathcal{L}^{\otimes n} \rightarrow \text{coker}(m_f) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_V \xrightarrow{m_g} \mathcal{O}_V \otimes \mathcal{L}^{\otimes n} \rightarrow \text{coker}(m_g) \rightarrow 0$$

where m_f, m_g are the obvious multiplication maps. By using the local isomorphisms $\mathcal{L}^{\otimes n}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ for some open cover $\{U_i\}_{i \in I}$, we can check that

$$\text{supp}(\text{coker}(m_f)) = V(f) \subset V$$

and

$$\text{supp}(\text{coker}(m_g)) = V(g) \subset V.$$

If we interpret the above exact sequences as distinguished triangles in $D^{\text{perf}}(X)_{(p+1)} / D^{\text{perf}}(X)_{(p)}$, we therefore see that both m_f and m_g are isomorphisms in this category, as

$$\text{codim}(V(f)) = \text{codim}(V(g)) = -p$$

which one deduces from looking at the local rings of the generic points of the irreducible components of $V(f), V(g)$ and applying Krull's principal ideal theorem. We can compose them to get an automorphism

$$\hat{h} : \mathcal{O}_V \xrightarrow{m_f} \mathcal{O}_V \otimes \mathcal{L}^{\otimes n} \xrightarrow{(m_g)^{-1}} \mathcal{O}_V$$

in the quotient category, where a corresponding fraction is given by the triangles associated to the two exact sequences above. It is now easy to check that we indeed have

$$\rho_X \left(\operatorname{div}^\Delta \left(\hat{h} \right) \right) = \operatorname{div}(h)$$

by using the explicit formula from the Remark 3.7.

4. FUNCTORIALITY

As we now have a reasonable definition of tensor-triangular Chow groups at hand, we would like to check that it has the functoriality properties one would expect it to have from the algebro-geometric Chow groups.

4.1. Functors with a relative dimension. We first have to define which class of functors we allow. In this section, \mathcal{K} and \mathcal{L} will always denote tensor-triangulated categories equipped with a dimension function.

Definition 4.1.1. Let $F : \mathcal{K} \rightarrow \mathcal{L}$ be an exact functor. We say that F has *relative dimension* n if there exists some $n \in \mathbb{Z}$ such that $F(\mathcal{K}_{(p)}) \subset \mathcal{L}_{(p+n)}$ for all p , and n is the smallest integer such that this relation holds.

Remark 4.1.2. Note that we *do not* require that F is a tensor functor (cf. Ex. 4.3). The composition of two functors of relative dimension n and m is a functor of relative dimension at most $n + m$. In all of the examples that follow, $n = 0$. However, the extra flexibility of having $n \neq 0$ might be useful for future applications.

Theorem 4.1.3. Let $F : \mathcal{K} \rightarrow \mathcal{L}$ be a functor of relative dimension n . Then F induces group homomorphisms

$$Z_p^\Delta(F) : Z_p^\Delta(\mathcal{K}) \rightarrow Z_{p+n}^\Delta(\mathcal{L}) \quad \text{and} \quad \operatorname{CH}_p^\Delta(F) : \operatorname{CH}_p^\Delta(\mathcal{K}) \rightarrow \operatorname{CH}_{p+n}^\Delta(\mathcal{L})$$

Proof. We have the following commutative diagram

$$\begin{array}{ccccc}
 \mathcal{K}_{(p)} & \xhookrightarrow{J_{\mathcal{K}}} & \mathcal{K}_{(p+1)} & \xrightarrow{F_{p+1}} & \mathcal{L}_{(p+n+1)} \\
 \downarrow Q_{\mathcal{K}} & \searrow F_p & & \searrow J_{\mathcal{L}} & \\
 \mathcal{K}_{(p)}/\mathcal{K}_{(p-1)} & & \mathcal{L}_{(p+n)} & \xhookrightarrow{J_{\mathcal{L}}} & \mathcal{L}_{(p+n+1)} \\
 \downarrow I_{\mathcal{K}} & \searrow \overline{F} & \downarrow Q_{\mathcal{L}} & & \\
 (\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^{\natural} & & \mathcal{L}_{(p+n)}/\mathcal{L}_{(p+n-1)} & & \\
 & \searrow \hat{F} & \downarrow I_{\mathcal{L}} & & \\
 & & (\mathcal{L}_{(p+n)}/\mathcal{L}_{(p+n-1)})^{\natural} & &
 \end{array}$$

where F_i is the restriction of F to $\mathcal{K}_{(i)}$ for $i = p, p+1$, \overline{F} exists because

$$F(\mathcal{K}_{p-1}) \subset \mathcal{L}_{p+n-1} = \ker(Q_{\mathcal{L}})$$

and \hat{F} exists as $I_{\mathcal{L}} \circ \overline{F}$ is a functor to an idempotent complete category. On applying the functor $K_0(-)$, we get the diagram

$$\begin{array}{ccccc}
K_0(\mathcal{K}_{(p)}) & \xrightarrow{j_{\mathcal{K}}} & K_0(\mathcal{K}_{(p+1)}) & & \\
\downarrow q_{\mathcal{K}} & \searrow f_p & & \searrow f_{p+1} & \\
K_0(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}) & & K_0(\mathcal{L}_{(p+n)}) & \xrightarrow{j_{\mathcal{L}}} & K_0(\mathcal{L}_{(p+n+1)}) \\
\downarrow i_{\mathcal{K}} & \searrow \bar{f} & \downarrow q_{\mathcal{L}} & & \\
K_0((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^{\natural}) & & K_0(\mathcal{L}_{(p+n)}/\mathcal{L}_{(p+n-1)}) & & \\
& \searrow \hat{f} & \downarrow i_{\mathcal{L}} & & \\
& & K_0((\mathcal{L}_{(p+n)}/\mathcal{L}_{(p+n-1)})^{\natural}) & &
\end{array}$$

where the lowercase arrows are induced by the corresponding uppercase ones. We set $Z_p^{\Delta}(F) := \hat{f}$. From the diagram, we also deduce that

$$\hat{f} \circ i_{\mathcal{K}} \circ q_{\mathcal{K}}(\ker(j_{\mathcal{K}})) \subset i_{\mathcal{L}} \circ q_{\mathcal{L}}(\ker(j_{\mathcal{L}}))$$

which implies that \hat{f} also induces a homomorphism $\text{CH}_p^{\Delta}(F)$ between the factor groups. \square

Remark 4.1.4. Theorem 4.1.3 and Remark 4.1.2 show that for all p , $Z_p^{\Delta}(-)$ and $\text{CH}_p^{\Delta}(-)$ are functors from the category of essentially small, rigid tensor-triangulated categories to the category of abelian groups, with respect to the class of functors with a relative dimension.

Remark 4.1.5. We can check that a functor $F : \mathcal{K} \rightarrow \mathcal{L}$ of relative dimension n also induces homomorphisms

$$\text{ch}_p^{\Delta}(F) : \text{ch}_p^{\Delta}(\mathcal{K}) \rightarrow \text{ch}_{p+n}^{\Delta}(\mathcal{L})$$

of our alternative Chow groups for all p . We have already seen in Theorem 4.1.3 how F induces a morphism of cycle groups, so it remains to check that F sends divisors to divisors. In order to do this, let $D = \text{div}(f)$ be the divisor of an automorphism $h : a \rightarrow a$ in $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$. Assume that a corresponding fraction for h is given by

$$a \xrightarrow{f} b \xleftarrow{g} a,$$

where $\text{cone}(f), \text{cone}(g) \in \mathcal{K}_{(p)}$.

As F has relative dimension n , we have $F(a) \in \mathcal{L}_{(p+n+1)}$ and $\text{cone}(F(f)), \text{cone}(F(g)) \in \mathcal{L}_{(p+n)}$, so there is an automorphism \hat{h} of $F(a)$ in $\mathcal{L}_{(p+n+1)}/\mathcal{L}_{(p+n)}$ given by the fraction

$$F(a) \xrightarrow{F(f)} F(b) \xleftarrow{F(g)} F(a)$$

and it makes sense to define $E := \text{div}(\hat{h})$. To see that $Z_p^{\Delta}(F)$ sends D to E , simply note that D corresponds to the element $[\text{cone}(f)] - [\text{cone}(g)] \in K_0(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})$ under the isomorphism

$$K_0(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}) \cong \coprod_{P \in \text{Spc}(\mathcal{K})_p} K_0(\text{Min}(\mathcal{K}_P))$$

and that

$$Z_p^{\Delta}(F)([\text{cone}(f)] - [\text{cone}(g)]) = [F(\text{cone}(f))] - [F(\text{cone}(g))] = [\text{cone}(F(f))] - [\text{cone}(F(g))].$$

We continue by giving two examples from algebraic geometry, which show that functors with a relative dimension occur naturally.

4.2. Example I: flat pullback. We fix X, Y separated, non-singular schemes of finite type over a field and $f : X \rightarrow Y$ a flat morphism. We consider $D^{\text{perf}}(X), D^{\text{perf}}(Y)$ with the standard structure of tensor-triangulated categories and assume they are equipped with the dimension function $-\text{codim}_{\text{Knull}}$.

Lemma 4.2.1. *The functor $Lf^* : D^{\text{perf}}(Y) \rightarrow D^{\text{perf}}(X)$ is of relative dimension 0.*

Proof. We need to check that for every complex $A^\bullet \in D^{\text{perf}}(Y)_{(p)}$, we have $Lf^*(A^\bullet) \in D^{\text{perf}}(X)_{(p)}$. Thus assume that

$$-\text{codim} \left(\text{supp} \left(\bigoplus_i H^i(A^\bullet) \right) \right) = q \leq p$$

As f is flat, f^* is exact, and so we have

$$\bigoplus_i H^i(Lf^*(A^\bullet)) = \bigoplus_i f^*(H^i(A^\bullet))$$

As the inverse image functor associated to a flat morphism preserves codimension of support (cf. [13, Section 1.3]), this implies that

$$-\text{codim} \left(\text{supp} \left(\bigoplus_i H^i(Lf^*(A^\bullet)) \right) \right) = -\text{codim} \left(\bigcup_i \text{supp}(f^*(H^i(A^\bullet))) \right) = q \leq p$$

as wanted. \square

Using the previous results, we know now that Lf^* induces morphisms between the tensor-triangular cycle and Chow groups of $D^{\text{perf}}(Y)$ and $D^{\text{perf}}(X)$. The following proposition shows that these are the expected ones.

Proposition 4.2.2. *Denote by $\rho_S : \text{CH}_p^\Delta(S) \rightarrow \text{CH}^{-p}(S)$ for $S = X, Y$ the isomorphisms from Theorem 3.4. Then for all p , there is a commutative diagram*

$$\begin{array}{ccc} \text{CH}_p^\Delta(D^{\text{perf}}(Y)) & \xrightarrow{\text{CH}_p^\Delta(Lf^*)} & \text{CH}_p^\Delta(D^{\text{perf}}(X)) \\ \downarrow \rho_Y & & \downarrow \rho_X \\ \text{CH}^{-p}(Y) & \xrightarrow{f^*} & \text{CH}^{-p}(X) \end{array}$$

where f^* denotes the flat pullback homomorphism on the usual Chow group. (cf. [14, Chapter 1.7]).

Proof. As both f^* and $\text{CH}_p^\Delta(Lf^*)$ are induced by the corresponding morphisms on the cycle level, it is enough to check that the diagram

$$\begin{array}{ccc} Z_p^\Delta(D^{\text{perf}}(Y)) & \xrightarrow{Z_p^\Delta(Lf^*)} & Z_p^\Delta(D^{\text{perf}}(X)) \\ \downarrow \rho_Y & & \downarrow \rho_X \\ Z^{-p}(Y) & \xrightarrow{f^*} & Z^{-p}(X) \end{array}$$

commutes. In order to do this, let $Z \subset Y$ be a subvariety (=reduced and irreducible subscheme) of Y of codimension $-p$, with associated ideal sheaf I_Z and cycle $[Z] \in Z^{-p}(Y)$. Consider the class $[W^\bullet]$ in

$$Z_p^\Delta(D^{\text{perf}}(Y)) = K_0(D^{\text{perf}}(Y)_{(p)} / D^{\text{perf}}(Y)_{(p-1)}) = K_0((D^{\text{perf}}(Y)_{(p)} / D^{\text{perf}}(Y)_{(p-1)})^\sharp)$$

where W^\bullet is the complex of sheaves concentrated in degree zero with $H^0(W^\bullet) = \mathcal{O}_Y/I_Z =: \mathcal{O}_Z$. It is not difficult to see that $\rho_Y([W^\bullet]) = Z \in Z^{-p}(Y)$. Indeed, using Remark 3.7 we calculate

$$\rho_Y([W^\bullet]) = \sum_i \sum_{P \in Y^{(-p)}} (-1)^i \text{length}_{\mathcal{O}_{Y,P}}(H^i(W^\bullet)_P) \cdot \overline{\{P\}}$$

where $H^i(W^\bullet)_P$ is the stalk of the i -th cohomology sheaf of the complex W^\bullet at the point P . Using that W^\bullet is concentrated in degree zero and that $\text{length}_{\mathcal{O}_{Y,P_Z}}(\mathcal{O}_{Z,P_Z}) = 1$, where P_Z is the generic point of Z , we see that $\rho_Y([W^\bullet]) = Z$.

Furthermore, using that f is flat, we compute that $Z_p^\Delta(Lf^*)([W^\bullet]) = [U^\bullet]$, where U^\bullet is the complex of sheaves concentrated in degree zero with $H^0(U^\bullet) = \mathcal{O}_X/I_{f^{-1}(Z)}$ and $f^{-1}(Z)$ denotes the scheme-theoretic inverse image of Z under f . Clearly, $\rho_X([U^\bullet]) = [f^{-1}(Z)]$, the cycle associated to the scheme-theoretic inverse image of Z and so we conclude that

$$\rho_X \circ Z_p^\Delta(Lf^*) \circ \rho_Y^{-1}([Z]) = [f^{-1}(Z)] = f^*[Z]$$

By additivity of the four maps in the diagram the proposition follows. \square

4.3. Example II: proper push-forward. We assume that X, Y are integral, non-singular, separated schemes of finite type over an algebraically closed field. (The latter assumption is needed in order to use [21, Proposition V.C.6.2]). Now, $f : X \rightarrow Y$ denotes a proper morphism. We consider $D^{\text{perf}}(X), D^{\text{perf}}(Y)$ with the standard structure of tensor-triangulated categories, *but this time we choose \dim_{Krull} as a dimension function*. Note that this implies $\text{CH}_p^\Delta(D^{\text{perf}}(S)) \cong \text{CH}^{\dim(S)-p}(S)$ for $S = X, Y$.

Lemma 4.3.1. *The functor $Rf_* : D^{\text{perf}}(X) \rightarrow D^{\text{perf}}(Y)$ has relative dimension 0.*

Proof. Let A^\bullet be a complex in $D^{\text{perf}}(X)$ such that $\dim(\text{supp}(\bigoplus_i H^i(A^\bullet))) \leq d$. There is a spectral sequence

$$E_2^{p,q} = R^p f_*(H^q(A^\bullet)) \implies H^{p+q}(Rf_*(A^\bullet))$$

(see for example [16, p.74 (3.4)]) that converges, as A^\bullet is bounded. By assumption, all the cohomology sheaves $H^q(A^\bullet)$ are supported in dimension $\leq d$, and by [21, Proposition V.C.6.2 (a)], we therefore have $\dim(\text{supp}(R^p f_*(H^q(A^\bullet)))) \leq d$ for all p . This implies that the terms $E_\infty^{p,q}$ are supported in dimension $\leq d$ as well. Therefore, all objects $H^{p+q}(Rf_*(A^\bullet))$ admit a finite filtration such that the subquotients are supported in dimension $\leq d$. An induction argument then shows that the same must hold for $H^{p+q}(Rf_*(a))$. We conclude that

$$\dim \left(\text{supp} \left(\bigoplus_i H^i(Rf_*(A^\bullet)) \right) \right) \leq d$$

which shows that $Rf_*(A^\bullet) \in D^{\text{perf}}(Y)_{(d)}$. To show that the relative dimension of Rf_* is 0, we need to show that there is a $B^\bullet \in D^{\text{perf}}(X)$ such that $\dim(\text{supp}(B^\bullet)) = \dim(\text{supp}(Rf_*(B^\bullet)))$. If P is any closed point of X with associated ideal sheaf I_P , then the complex C_P^\bullet concentrated in degree 0 with \mathcal{O}_X/I_P has $\dim(\text{supp}(C_P^\bullet)) = 0$. By the result we just proved, $Rf_*(C_P^\bullet) \in D^{\text{perf}}(Y)_{(0)}$, which implies that either $Rf_*(C_P^\bullet) = 0$ or $\dim(\text{supp}(Rf_*(C_P^\bullet))) = 0$. If $Rf_*(C_P^\bullet) = 0$, we would certainly have $H^0(Rf_*(C_P^\bullet)) = 0$, but this is impossible by the spectral sequence we used above: indeed, it is easy to see that $E_\infty^{0,0} = E_2^{0,0}$ as $H^i(C_P^\bullet) = 0$ for $i \neq 0$. But we have $E_2^{0,0} = R^0 f_*(C_P^\bullet) = f_*(\mathcal{O}_X/I_P) \neq 0$. Thus $H^0(Rf_*(C_P^\bullet))$ has a non-zero subquotient which implies that $Rf_*(C_P^\bullet) \neq 0$. We conclude that $\dim(\text{supp}(Rf_*(C_P^\bullet))) = 0$ which completes the proof. \square

The previous lemma establishes that Rf_* induces homomorphisms

$$\text{CH}_p^\Delta(D^{\text{perf}}(X)) \rightarrow \text{CH}_p^\Delta(D^{\text{perf}}(Y))$$

for all p . Again, we can show that these are exactly the ones we would expect.

Proposition 4.3.2. *Denote by $\rho_S : \mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(S)) \rightarrow \mathrm{CH}^{\dim(S)-p}(S)$ for $S = X, Y$ the isomorphisms from Theorem 3.4. Then for all p , there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(X)) & \xrightarrow{\mathrm{CH}_p^\Delta(Rf_*)} & \mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(Y)) \\ \downarrow \rho_X & & \downarrow \rho_Y \\ \mathrm{CH}^{\dim(X)-p}(X) & \xrightarrow{f_*} & \mathrm{CH}^{\dim(Y)-p}(Y) \end{array}$$

where f_* denotes the proper push-forward homomorphism on the usual Chow group (cf. [14, Chapter 1.4]).

Proof. Again, it suffices to show the statement for the maps on the cycle groups, as the maps on the Chow groups are induced by those. By additivity of the four maps in the diagram it is enough to check that for an (integral) subvariety $V \subset X$ of dimension p and an element $v \in Z_p^\Delta(\mathrm{D}^b(X))$ with $\rho_X(v) = [V]$, we have $\rho_Y \circ Z_p^\Delta(Rf_*)(v) = f_*([V])$. So, fix V as above and consider the complex of coherent sheaves W^\bullet that is concentrated in degree 0 and has $H^0(W^\bullet) = \mathcal{O}_V$, where $\mathcal{O}_V = \mathcal{O}_X/\mathcal{I}_V$ and \mathcal{I}_V is the ideal sheaf associated to V . The complex W^\bullet represents a class $[W^\bullet]$ in

$$Z_p^\Delta(\mathrm{D}^b(X)) = K_0(\mathrm{D}^b(X)_{(p)}/\mathrm{D}^b(X)_{(p-1)}) = K_0\left((\mathrm{D}^b(X)_{(p)}/\mathrm{D}^b(X)_{(p-1)})^{\mathbb{Q}}\right)$$

and similarly to the calculation in Proposition 4.2.2 we see that $\rho_X([W^\bullet]) = V$.

For the next step, we compute

$$\begin{aligned} \rho_Y \circ Z_p^\Delta(Rf_*)([W^\bullet]) &= \sum_i \sum_{Q \in Y_{(p)}} (-1)^i \mathrm{length}_{\mathcal{O}_{Y,Q}}(H^i(Rf_*(W^\bullet))_Q) \cdot \overline{\{Q\}} \\ &= \sum_i \sum_{Q \in Y_{(p)}} (-1)^i \mathrm{length}_{\mathcal{O}_{Y,Q}}(R^i f_*(\mathcal{O}_V)_Q) \cdot \overline{\{Q\}} \\ &= \sum_i (-1)^i \sum_{Q \in Y_{(p)}} \mathrm{length}_{\mathcal{O}_{Y,Q}}(R^i f_*(\mathcal{O}_V)_Q) \cdot \overline{\{Q\}}. \end{aligned}$$

Using [21, Proposition V.C.6.2 (b)], we see that this is equal to $f_*(V)$, which means that we have shown $\rho_Y \circ Z_p^\Delta(Rf_*)([W^\bullet]) = f_*(V)$ and thus have finished the proof of the theorem. \square

5. SOME COMPUTATIONS IN MODULAR REPRESENTATION THEORY

So far we have only looked at examples from algebraic geometry. However, tensor-triangulated categories also occur in different contexts. One of these is modular representation theory, where one studies kG -modules for a finite group G and a field k such that $\mathrm{char}(k)$ divides $|G|$. A useful tool in this context is the stable category $kG\text{-stab}$, which is obtained as the quotient of $kG\text{-mod}$ by the subcategory of projective kG -modules and is a tensor-triangulated category. Using the theory from the previous sections, we therefore have a notion of tensor-triangular Chow groups for these categories. In this section we compute concrete examples of these groups.

5.1. Basic definitions and results. We recall some basic definitions and results that we will need. All of them can be found in the books by Carlson [11] and Benson [8, 9] or in Balmer's article [5]. For the rest of the section, G will denote a finite group, k is a field of characteristic p dividing $|G|$, and kG is the corresponding group algebra. Associated to this algebra is the abelian category $kG\text{-mod}$ consisting of the finitely-generated left kG -modules. It is a Frobenius category (i.e. an exact category in the sense of Quillen, with enough injectives and projectives and where

the classes of projective and injective objects coincide), and so we can form the stable category $kG\text{-stab}$ which is naturally triangulated. It can be given a symmetric-monoidal structure with the tensor product $- \otimes_k -$ with unit object k , the trivial kG -module. One also checks that $kG\text{-stab}$ is rigid, where the dual of an object M is given as $\text{Hom}_k(M, k)$.

Definition 5.1.1. The *cohomology ring* of kG is defined as the graded ring

$$H^*(G, k) := \bigoplus_{i \geq 0} \text{Ext}_{kG}^i(k, k)$$

The *projective support variety* of kG is defined as

$$\mathcal{V}_G(k) := \text{Proj}(H^*(G, k))$$

Remark 5.1.2. When p is odd, $H^*(G, k)$ is in general only a *graded* commutative ring, so when we write $\text{Proj}(H^*(G, k))$ we really mean $\text{Proj}(H^{\text{ev}}(G, k))$ in this case, where $H^{\text{ev}}(G, k)$ is the subring of all elements of even degree. Another way to deal with this difficulty is to define a version of Proj for graded-commutative k -algebras in which case the result still holds (cf. [2, Section 1]).

For any two finite-dimensional kG -modules M, N , $\bigoplus_{i \geq 0} \text{Ext}_{kG}^i(M, N)$ is a finitely generated module over $H^*(G, k)$ by the Evens-Venkov theorem, cf. [11, Theorem 9.1].

Definition 5.1.3. For a kG -module $M \neq 0$, define $J(M) \subset H^*(G, k)$ as the annihilator ideal of $\text{Ext}_{kG}^*(M, M)$ in $H^*(G, k)$. The *variety* of M is the subvariety of $\mathcal{V}_G(k)$ associated to $J(M)$.

Definition 5.1.4. Let M be in $kG\text{-mod}$ and let $P^\bullet \rightarrow M$ be a minimal projective resolution. The *complexity* $c_G(M)$ of M is defined as the least integer s such that there is a constant $\kappa > 0$ with

$$\dim_k(P^n) \leq \kappa \cdot n^{s-1} \quad \text{for } n > 0$$

The complexity of a module can be read off from its variety:

Theorem 5.1.5 (cf. [8, Prop. 5.7.2]). *If M is a finitely generated kG -module, then*

$$\dim(\mathcal{V}_G(M)) = c_G(M) - 1.$$

The projective support variety of kG can be reconstructed from $kG\text{-stab}$:

Theorem 5.1.6 (cf. [5, Corollary 5.10]). *There is a homeomorphism*

$$\rho : \text{Spc}(kG\text{-stab}) \longrightarrow \mathcal{V}_G(k).$$

Furthermore, the support of a module $M \in kG\text{-stab}$ corresponds to $\mathcal{V}_G(M)$ under this map.

From now on, we will take \dim_{Krull} (c.f. Example 2.5) as a dimension function for $kG\text{-stab}$. By Theorem 5.1.6 this coincides with the usual Krull dimension on $\mathcal{V}_G(k)$ under the homeomorphism ρ .

With these results at hand, we proceed to compute some examples of tensor-triangular Chow groups coming from $kG\text{-stab}$.

5.2. The case $G = \mathbb{Z}/p^n\mathbb{Z}$. We begin with the case where $G = \mathbb{Z}/p^n\mathbb{Z}$ for some prime p and $n \in \mathbb{N}$. In the following, k will be any field of characteristic p . It follows from [11, Theorem 7.3] that $\mathcal{V}_G(k)$ is a point, and so a finitely generated kG -module has complexity 1 if and only if it is non-projective.

Computing the tensor-triangular cycle groups for $kG\text{-stab}$ amounts to calculating

$$K_0((kG\text{-stab}_{(i)} / kG\text{-stab}_{(i-1)})^{\natural})$$

One immediately sees that the only non-trivial case is when $i = 0$. Then $Z_0^\Delta(kG\text{-stab}) = K_0(kG\text{-stab})$, as $kG\text{-stab}$ is idempotent complete. In order to compute this, note that $K_0(kG\text{-mod}) \cong$

\mathbb{Z} , as kG is a (commutative) local artinian ring. For local rings, projective and free modules coincide, and thus it follows from [25, Proposition 1] that $Z_0^\Delta(kG\text{-stab}) \cong \mathbb{Z}/p^n\mathbb{Z}$. We also see that this group coincides with $\text{CH}_0^\Delta(kG\text{-stab})$, as we are in the top dimension. Summarising, we have the following:

Proposition 5.2.1. *Let $G = \mathbb{Z}/p^n\mathbb{Z}$ for some prime p and $n \in \mathbb{N}$ and k any field of characteristic p . Then*

$$Z_i^\Delta(kG\text{-stab}) = \text{CH}_i^\Delta(kG\text{-stab}) = 0 \quad \text{for all } i \neq 0$$

and

$$Z_0^\Delta(kG\text{-stab}) = \text{CH}_0^\Delta(kG\text{-stab}) = \mathbb{Z}/p^n\mathbb{Z}.$$

5.3. The case $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y | x^2 = y^2 = 1, xy = yx \rangle$ and k is a field of characteristic 2, the computations become more involved.

As a consequence of [11, Theorem 7.6], we have that $\mathcal{V}_G(k) = \mathbb{P}^1$. Therefore there is a proper subcategory of $kG\text{-stab}$ consisting of the modules of complexity ≤ 1 . In order to work with those, we need the following classification:

Lemma 5.3.1. *All finite-dimensional indecomposable kG -modules of odd dimension have complexity 2.*

Proof. Let M be a odd-dimensional indecomposable module. If we assume that M has complexity 1, then by [8, Theorem 5.10.4 and Corollary 5.10.7], M must be periodic, with period 1. In other words, if $\epsilon : P \twoheadrightarrow M$ is a projective cover of M , then we must have $M \cong \ker(\epsilon)$. However, kG is a local ring, and so the only indecomposable projective module is the free module of rank 1, which has k -dimension 4. Thus, if M has dimension $2n + 1$ and P has dimension $4m$, then using that ϵ is surjective and the dimension formula, we get $\dim_k(\ker(\epsilon)) = 4m - 2n - 1$. We see immediately that $\ker(\epsilon)$ cannot have dimension $2n + 1$, and thus M cannot have complexity 1. As it is non-projective it therefore must have complexity 2. \square

We also see that a complementary result holds for the even-dimensional representations:

Lemma 5.3.2. *All finite-dimensional, non-projective indecomposable kG -modules of even dimension have complexity 1.*

Proof. It follows from [12, Proposition 3.1] that a non-projective indecomposable kG -module of even dimension is periodic with period 1. As an immediate consequence, those modules have complexity 1. \square

Remark 5.3.3. Lemma 5.3.2 also follows from the following explicit calculation: using the classification of all indecomposable kG -modules (cf. [9, Theorem 4.3.3]), one sees that any non-projective, indecomposable even-dimensional kG -module is isomorphic to one of the form

$$x \mapsto \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \quad y \mapsto \begin{pmatrix} I & J \\ 0 & I \end{pmatrix}$$

where I is the $n \times n$ identity matrix and J is some $n \times n$ matrix over k . Note that in this presentation, the above modules may fail to be mutually non-isomorphic for different J . This type of module will from now on be denoted by $M_n(J)$ and we proceed to find the first term of a projective resolution for it. In order to do so, fix the basis $(1, x + 1, y + 1, xy + x + y + 1)$ for kG and consider for $1 \leq i \leq n$ the $4 \times n$ matrices

$$B_i := \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & f_i & J_i & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \text{and} \quad E_i := \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ f_i & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where f_i is the i -th standard basis vector of length n and J_i is the i -th column vector of J . One now verifies the following statement by an explicit computation:

Proposition 5.3.4. *The linear map $\epsilon : kG^n \rightarrow M_n(J)$ given by the $4n \times 2n$ matrix*

$$\begin{pmatrix} B_1 & \cdots & B_n \\ E_1 & \cdots & E_n \end{pmatrix}$$

is a surjective kG -module homomorphism. Furthermore we have $\ker(\epsilon) \cong M_n(J)$. \square

From this it follows that the complexity of $M_n(J)$ is ≤ 1 . As it is not projective, it must therefore have complexity 1.

The following statement is an immediate consequence of Lemma 5.3.1 and Lemma 5.3.2:

Corollary 5.3.5. *The indecomposable kG -modules of odd dimension are exactly the indecomposable modules of complexity 2. The non-projective indecomposable kG -modules of even dimension are exactly the indecomposable modules of complexity 1.* \square

Using this classification, we can calculate the zero-dimensional Chow group:

Proposition 5.3.6. *There is an isomorphism*

$$\mathrm{CH}_0^\Delta(kG\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. By definition, $\mathrm{Z}_0^\Delta(kG\text{-stab}) = K_0((kG\text{-stab}_{(0)})^\natural) \cong K_0(kG\text{-stab}_{(0)})$, as $kG\text{-stab}_{(-1)} = 0$ and thick subcategories of idempotent complete categories are idempotent complete themselves. Using this, we have that $\mathrm{CH}_0^\Delta(kG\text{-stab}) \cong \mathrm{Z}_0^\Delta(kG\text{-stab}) / \ker(\alpha)$, where

$$\alpha : K_0(kG\text{-stab}_{(0)}) \rightarrow K_0(kG\text{-stab}_{(1)}) = K_0(kG\text{-stab})$$

is induced by the inclusion $kG\text{-stab}_{(0)} \hookrightarrow kG\text{-stab}$. Using the isomorphism theorem for abelian groups, we conclude that $\mathrm{CH}_0^\Delta(kG\text{-stab}) \cong \mathrm{im}(\alpha)$. Using that kG is a local ring (which has a unique simple module) and [25, Proposition 1] again, we see that $K_0(kG\text{-stab}) \cong \mathbb{Z}/4\mathbb{Z}$. From Corollary 5.3.5 we know that $kG\text{-stab}_{(0)}$ consists of modules that are direct sums of even-dimensional kG -modules. Thus, their image in $K_0(kG\text{-stab})$ is exactly $\mathbb{Z}/2\mathbb{Z}$. \square

For the one-dimensional Chow group we need to work a bit harder. We first take a closer look at the quotient $\mathcal{L} := kG\text{-stab} / kG\text{-stab}_{(0)}$.

Lemma 5.3.7. *Assume k is algebraically closed. The category \mathcal{L} is idempotent complete.*

Proof. Under the additional hypothesis, it is shown in [10, Example 5.1] that up to isomorphism, the only indecomposable object in \mathcal{L} is k , which has endomorphism ring $K := k(\zeta)$, a transcendental field extension of k . It follows that \mathcal{L} is equivalent to the category of finite-dimensional vector spaces over K , which is idempotent complete. \square

This enables us to prove the following:

Proposition 5.3.8. *Assume k is algebraically closed. There is an isomorphism*

$$\mathrm{CH}_1^\Delta(kG\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. The sequence of triangulated categories

$$kG\text{-stab}_{(0)} \hookrightarrow kG\text{-stab} \rightarrow kG\text{-stab}/kG\text{-stab}_{(0)}$$

induces an exact sequence

$$K_0(kG\text{-stab}_{(0)}) \xrightarrow{\alpha} K_0(kG\text{-stab}) \rightarrow K_0(kG\text{-stab}/kG\text{-stab}_{(0)}) \rightarrow 0$$

where α is the same as in the proof of Proposition 5.3.6. Therefore,

$$\mathrm{CH}_1^\Delta(kG\text{-stab}) \cong K_0((kG\text{-stab}/kG\text{-stab}_{(0)})^\sharp) \cong K_0(kG\text{-stab})/\mathrm{im}(\alpha) \cong \mathbb{Z}/2\mathbb{Z}$$

as follows from Lemma 5.3.7 and the proof of Proposition 5.3.6. \square

6. RELATIVE TENSOR-TRIANGULAR CHOW GROUPS AND AGREEMENT FOR $D_{\mathrm{QCOH}}(X)$

So far we have considered tensor-triangular Chow groups only for essentially small tensor-triangulated categories. For tensor-triangulated categories that are not essentially small we run into problems: for example, for such categories \mathcal{T} we have no definition of $\mathrm{Spc}(\mathcal{T})$, so it does not even make sense to define subcategories like $\mathcal{T}_{(p)}$. The situation changes when we assume that \mathcal{T} is compactly rigidly generated, i.e. the compact objects $\mathcal{T}^c \subset \mathcal{T}$ form a set, coincide with the rigid ones, and they generate \mathcal{T} . In this case, Balmer and Favi show in [3] that there is a notion of support for objects of \mathcal{T} which assigns to an object $A \in \mathcal{T}$ a (not necessarily closed) subset $\mathrm{supp}(A) \subset \mathrm{Spc}(\mathcal{T}^c)$. This is generalised by Stevenson's work [23], which introduces the concept of an action of a compactly-rigidly generated tensor-triangulated category \mathcal{T} on a triangulated category \mathcal{K} (which need not have a symmetric monoidal structure). In this setting it is possible to define a notion of relative support for objects of \mathcal{K} , which assigns to an object $A \in \mathcal{K}$ a subset $\mathrm{supp}(A) \subset \mathrm{Spc}(\mathcal{T}^c)$. It recovers the notion of [3] mentioned before, when we set $\mathcal{K} = \mathcal{T}$ and act via the tensor product of \mathcal{T} . This construction is the starting point for our definition of tensor-triangular Chow groups for \mathcal{K} , relative to the action of \mathcal{T} .

6.1. Preliminaries. In this section, \mathcal{T} will denote a triangulated category with compatible symmetric monoidal structure. We assume that it is compactly-rigidly generated and that it has arbitrary set-indexed coproducts. Note that this implies that \mathcal{T} is not essentially small. The subcategory of compact objects of \mathcal{T} will be denoted by \mathcal{T}^c . The category \mathcal{T}^c is a tensor-triangulated category in the sense of Convention 2.1 [3, Hypotheses 1.1]. We will assume that $\mathrm{Spc}(\mathcal{T}^c)$ is a noetherian topological space and that \mathcal{T}^c is equipped with a dimension function \dim . The category \mathcal{T} acts on a triangulated category \mathcal{K} via an action $*$ in the sense of [23]. The category \mathcal{K} is assumed to be compactly generated and to have arbitrary coproducts as well.

Following [3], we can assign to every object $a \in \mathcal{T}$ a support $\mathrm{supp}(a) \subset \mathrm{Spc}(\mathcal{T}^c)$: Given a specialisation-closed subset $V \subset \mathrm{Spc}(\mathcal{T}^c)$, we have a distinguished triangle

$$\Gamma_V(\mathbb{I}) \rightarrow \mathbb{I} \rightarrow L_V(\mathbb{I}) \rightarrow \Sigma \Gamma_V(\mathbb{I})$$

where Γ_V and L_V are the acyclisation and localisation functors associated to the smashing ideal generated by the compact objects with support in V . For objects $A \in \mathcal{K}$, we set $\Gamma_V A := \Gamma_V(\mathbb{I}) * A$ and $L_V A := L_V(\mathbb{I}) * A$.

For a point $x \in \mathrm{Spc}(\mathcal{T}^c)$ the subsets $\overline{\{x\}}$ and $Y_x := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) : x \not\subseteq \mathcal{P}\}$ are specialisation-closed and so we define the “residue object” $\Gamma_x \mathbb{I} \in \mathcal{T}$ as $\Gamma_{\overline{\{x\}}} L_{Y_x}(\mathbb{I})$. For $A \in \mathcal{T}$, we now define

$$\mathrm{supp}(A) := \{x \in \mathrm{Spc}(\mathcal{T}^c) | \Gamma_x \mathbb{I} \otimes A \neq 0\}.$$

In [23], the same residue objects are used to define supports for objects of \mathcal{K} : for an object $B \in \mathcal{K}$, we set

$$\mathrm{supp}(B) := \{x \in \mathrm{Spc}(\mathcal{T}^c) | \Gamma_x A := \Gamma_x \mathbb{I} * A \neq 0\}.$$

This gives rise to order-preserving maps

$$\begin{aligned} \{\text{subsets of } \text{Spc}(\mathcal{T}^c) \cap \text{supp}(\mathcal{K})\} &\xrightleftharpoons[\sigma_{\mathcal{K}}]{\tau_{\mathcal{K}}} \{\text{localising } \mathcal{T}\text{-submodules of } \mathcal{K}\} \\ S &\mapsto \{t \in \mathcal{K} : \text{supp}(t) \subset S\} \\ \bigcup_{t \in I} \text{supp}(t) &\hookleftarrow I \end{aligned}$$

where the ordering on both sides is given by inclusion.

Definition 6.1.1 (cf. [23, Definition 6.1]). We say that the action $*$ of \mathcal{T} on \mathcal{K} satisfies the *local-to-global principle* if for each A in \mathcal{K}

$$\langle A \rangle_* = \langle \Gamma_x A | x \in \text{Spc}(\mathcal{T}^c) \rangle_*$$

where for a collection of objects $S \subset \mathcal{K}$ we denote by $\langle S \rangle_*$ the smallest localising \mathcal{T} -submodule of \mathcal{K} containing S .

Remark 6.1.2. The local-to-global principle holds very often, e.g. when \mathcal{T} arises as the homotopy category of a monoidal model category (cf. [23, Proposition 6.8]). If the local-to-global principle holds, it has the useful consequence that supp detects the vanishing of an object: if $\text{supp}(A) = \emptyset$ then

$$\langle A \rangle_* = \langle \Gamma_x A | x \in \text{Spc}(\mathcal{T}^c) \rangle_* = \langle 0 \rangle_* = 0$$

which implies that $A = 0$.

For $p \in \mathbb{Z}$, let

$$V_{\leq p} := \{x \in \text{Spc}(\mathcal{T}^c) | \dim(x) \leq p\}, V_p := \{x \in \text{Spc}(\mathcal{T}^c) | \dim(x) = p\}$$

and set $\Gamma_p A := \Gamma_{V_{\leq p}} L_{V_{\leq p-1}} A$. In [24], the following decomposition theorem, analogous to the one of [6], is proved:

Proposition 6.1.3 (cf. [24, Lemma 4.3]). *Suppose that the action of \mathcal{T} on \mathcal{K} satisfies the local-to-global principle and let $p \in \mathbb{Z}$. There is an equality of subcategories*

$$\Gamma_p \mathcal{K} = \tau_{\mathcal{K}}(V_p) = \prod_{x \in V_p} \Gamma_x \mathcal{K}$$

where $\Gamma_x \mathcal{K}$ denotes the essential image of the functor $\Gamma_x(\mathbb{I}) * -$. □

We give another description of $\Gamma_p \mathcal{K}$ that bears more resemblance to what we have seen for essentially small tensor-triangulated categories. For $p \in \mathbb{Z}$, define $\mathcal{K}_{(p)} := \tau_{\mathcal{K}}(V_{\leq p})$.

Lemma 6.1.4. *Assume the local-to-global principle holds for the action of \mathcal{T} on \mathcal{K} . Then, for all $p \in \mathbb{Z}$, there is an equality of subcategories*

$$\mathcal{K}_{(p)} = \Gamma_{V_{\leq p}} \mathcal{K} = \{A \in \mathcal{K} | \exists A' : A \cong \Gamma_{V_{\leq p}}(\mathbb{I}) * A'\}$$

Proof. Let $A \in \Gamma_{V_{\leq p}} \mathcal{K}$, then we have an isomorphism $A \cong \Gamma_{V_{\leq p}}(\mathbb{I}) * A'$. By [23, Proposition 5.5], we know that $\text{supp}(A) = \text{supp}(\Gamma_{V_{\leq p}}(\mathbb{I}) * A') = \text{supp}(A') \cap V_{\leq p}$, from which it follows that A is supported in dimension $\leq p$. Thus, $A \in \mathcal{K}_{(p)}$.

Conversely, assume that $A \in \mathcal{K}_{(p)}$. We apply the functor $- * A$ to the localisation triangle

$$\Gamma_{V_{\leq p}}(\mathbb{I}) \rightarrow \mathbb{I} \rightarrow L_{V_{\leq p}}(\mathbb{I}) \rightarrow \Sigma \Gamma_{V_{\leq p}}(\mathbb{I})$$

to obtain the triangle

$$\Gamma_{V_{\leq p}}(\mathbb{I}) * A \rightarrow A \rightarrow L_{V_{\leq p}}(\mathbb{I}) * A \rightarrow \Sigma \Gamma_{V_{\leq p}}(\mathbb{I}) * A$$

As A is supported in dimension $\leq p$, it follows again from [23, Proposition 5.5] that

$$\operatorname{supp}(L_{V_{\leq p}}(\mathbb{I}) * A) = \emptyset$$

and therefore $L_{V_{\leq p}}(\mathbb{I}) * A = 0$ by the local-to-global principle, as we saw in Remark 6.1.2. Therefore $\Gamma_{V_{\leq p}}(\mathbb{I}) * A \cong A$ which implies that $A \in \Gamma_{V_{\leq p}}\mathcal{K}$. \square

The following statement is the desired description of $\Gamma_p\mathcal{K}$. Its proof is very similar to that of Lemma 6.1.4.

Lemma 6.1.5. *Suppose that the action of \mathcal{T} on \mathcal{K} satisfies the local-to-global principle and let $p \in \mathbb{Z}$. There is an equality of subcategories*

$$\Gamma_p\mathcal{K} = \mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$$

where we view the latter quotient as the essential image of the localisation functor $L_{V_{\leq p-1}}(\mathbb{I}) * -$ restricted to $\mathcal{K}_{(p)}$. \square

Remark 6.1.6. We can push the analogy with the decomposition theorem of [6] even further: for $x \in \operatorname{Spc}(\mathcal{T}^c)$, define \mathcal{K}_x as the essential image of the localisation functor $L_{\langle x \rangle}\mathbb{I} * -$ associated to localising subcategory $\langle x \rangle \subset \mathcal{T}$ and denote by $\operatorname{Min}(\mathcal{K}_x)$ the subcategory of objects with minimal support. Then, similarly to the proofs of Lemma 6.1.4 and 6.1.5, one can show that there is an equality of subcategories

$$\operatorname{Min}(\mathcal{K}_x) = \tau_{\mathcal{K}}(\{x\}) = \Gamma_x\mathcal{K}.$$

Proposition 6.1.3 and Lemma 6.1.5 serve as a motivation for the definition of relative tensor-triangular Chow groups, in the same way that the decomposition theorem from [6] motivated Definitions 2.7 and 2.8.

6.2. Relative tensor-triangular Chow groups. From now on, we will assume that the local-to-global principle holds for the action of \mathcal{T} on \mathcal{K} . As $V_{\leq p}$ is specialisation-closed for all $p \in \mathbb{Z}$, it follows from [23, Corollary 4.11] that $\mathcal{K}_{(p)} = \Gamma_{V_{\leq p}}\mathcal{K}$ is compactly generated for all $p \in \mathbb{Z}$. By [18, Theorem 5.6.1] we therefore have that $(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c$ is the thick closure of $\mathcal{K}_{(p)}^c/\mathcal{K}_{(p-1)}^c$ in $\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$ (cf. [3, Theorem 4.1]). Therefore, we get an injection

$$j : K_0\left(\mathcal{K}_{(p)}^c/\mathcal{K}_{(p-1)}^c\right) \hookrightarrow K_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c\right).$$

Note that it is necessary to restrict to compact objects in order to get something non-trivial, as $\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$ has arbitrary coproducts and thus trivial Grothendieck group. Furthermore, the quotient functor $\mathcal{K}_{(p)}^c \rightarrow \mathcal{K}_{(p)}^c/\mathcal{K}_{(p-1)}^c$ and the embedding $\mathcal{K}_{(p)}^c \rightarrow \mathcal{K}_{(p+1)}^c$ induce maps

$$q : K_0\left(\mathcal{K}_{(p)}^c\right) \rightarrow K_0\left(\mathcal{K}_{(p)}^c/\mathcal{K}_{(p-1)}^c\right) \quad \text{and} \quad i : K_0\left(\mathcal{K}_{(p)}^c\right) \rightarrow K_0\left(\mathcal{K}_{(p+1)}^c\right).$$

The decomposition theorem 6.1.3 now motivates the following definition:

Definition 6.2.1. We define the p -dimensional tensor-triangular cycle groups respectively p -dimensional tensor-triangular Chow groups of \mathcal{K} , relative to the action of \mathcal{T} as follows:

$$Z_p^\Delta(\mathcal{T}, \mathcal{K}) := K_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c\right) \quad \text{and} \quad \operatorname{CH}_p^\Delta(\mathcal{T}, \mathcal{K}) := Z_p^\Delta(\mathcal{T}, \mathcal{K})/j \circ q(\ker(i)).$$

Remark 6.2.2. As we have assumed that the local-to-global principle is satisfied, we can view an element of $Z_p^\Delta(\mathcal{T}, \mathcal{K})$ as a formal sum of p -dimensional points x_i of $\operatorname{Spc}(\mathcal{T}^c)$, with coefficients in $K_0((\Gamma_x\mathcal{K})^c)$ for $x \in V_p$, by Proposition 6.1.3 and Lemma 6.1.5.

Proposition 6.2.3. *Consider the action of \mathcal{T} on itself via its tensor product and assume that the local-to-global principle holds for this action. Then we have isomorphisms*

$$Z_p^\Delta(\mathcal{T}, \mathcal{T}) \cong Z_p^\Delta(\mathcal{T}^c) \quad \text{and} \quad \mathrm{CH}_p^\Delta(\mathcal{T}, \mathcal{T}) \cong \mathrm{CH}_p^\Delta(\mathcal{T}^c).$$

Proof. By definition, $Z_p^\Delta(\mathcal{T}, \mathcal{T}) = K_0((\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)})^c)$. As mentioned before, it follows from [18, Theorem 5.6.1] that $(\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)})^c$ is the thick closure of $\mathcal{T}_{(p)}^c/\mathcal{T}_{(p-1)}^c$ in $\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)}$. But the latter category is idempotent complete as it has arbitrary coproducts (cf. [19, Proposition 1.6.8]), and so $(\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)})^c$ is equivalent to $(\mathcal{T}_{(p)}^c/\mathcal{T}_{(p-1)}^c)^\natural$. We conclude that

$$Z_p^\Delta(\mathcal{T}, \mathcal{T}) = K_0((\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)})^c) \cong K_0\left(\left(\mathcal{T}_{(p)}^c/\mathcal{T}_{(p-1)}^c\right)^\natural\right)$$

which is equal to $Z_p^\Delta(\mathcal{T}^c)$ by definition. Checking that the notions of rational equivalence agree in this case is now easy. \square

Now, let X be a noetherian scheme and denote by $D(X) := D_{\mathrm{Qcoh}}(X)$ the full derived category of complexes of \mathcal{O}_X -modules with quasi-coherent cohomology. The category $D(X)$ is a compactly-rigidly generated tensor-triangulated category with arbitrary coproducts, and we have $D(X)^c = D^{\mathrm{perf}}(X)$ (cf. [3, Example 1.2 (2)]).

Corollary 6.2.4. *We have isomorphisms*

$$Z_p^\Delta(D(X), D(X)) \cong Z_p^\Delta(D^{\mathrm{perf}}(X)) \quad \text{and} \quad \mathrm{CH}_p^\Delta(D(X), D(X)) \cong \mathrm{CH}_p^\Delta(D^{\mathrm{perf}}(X)).$$

In particular, if X is separated, non-singular of finite type over a field and we equip $D^{\mathrm{perf}}(X)$ with the opposite of the Krull codimension as a dimension function, we have

$$Z_p^\Delta(D(X), D(X)) \cong Z^{-p}(X) \quad \text{and} \quad \mathrm{CH}_p^\Delta(D(X), D(X)) \cong \mathrm{CH}^{-p}(X).$$

Proof. This is an immediate consequence of Proposition 6.2.3, the fact that the local-to-global principle holds for the action of $D(X)$ on itself and Theorem 3.4. \square

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